ECS5351-Multidimensional Signal Analysis Notes

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1 Chapter1

1.1 Tuesday 8/27/2024

1.1.1 Signal Classes

- **periodic** classes:
$$x(t) = x(t - \tau), \tau > 0 \tag{1}$$
- **finite energy** signals:
$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \tag{2}$$
Why is this considered energy? know power is $p(t) = |i(t)|^2$, so $E = \int p(t) = \int |i(t)|^2 dt$, otherwise known as the energy dissipated by a resistor
- **finite area** signals:
$$E = \int |x(t)| dt < \infty \tag{3}$$
- **bounded**
- **band limited** (don't exist in reality)
- **analytic**: if a taylor series converges everywhere note: all band limited signals are analytic
- **causal**:
$$x(t) = 0, \forall t < 0 \tag{4}$$
- **even/odd** signals note: can always represent a signal in terms of even and odd parts $c(t) = e(t) + o(t)$, $e(t) = \frac{1}{2}(c(t) + c(-t))o(t) = \frac{1}{2}(c(t) - c(-t))$

1.2 Thursday 8/29/2024

1.2.1 Fourier Transform

Definition 1 (Fourier Transform). *For a continuous function $x(t)$, the fourier transform is:*

$$X(u) = \int x(t) e^{-j2\pi ut} dt$$
$$x(t) \leftrightarrow X(u)$$

We will use capital letters to denote the fourier dual of a function/signal. Example of units: volts $\rightarrow$ volts seconds **Inverse FT** definition:

$$x(t) = \int X(u)e^{j2\pi ut}dt \tag{5}$$

Definition 2 (Fourier Series)**.** *Assuming x(t) is period,*

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n \frac{t}{T}}$$

$$c_n = \frac{1}{T}\int x(t)e^{-j2\pi n \frac{t}{T}}dt$$

note: $\int_0^T := \int_{-\tau}^{\tau+T}$ due to periodicity if $x_\tau(x)$ period of $x(t)$, then $c_n = \frac{1}{T}\int x(t)e^{-j2\pi n\frac{t}{T}}dt = \frac{1}{T}\int_{-\infty}^{\infty} x_\tau(t)e^{-j2\pi n\frac{t}{T}}dt = \frac{1}{T}X_T(\frac{n}{T})$

It's easiest to compute fourier signals: take the FT of a signal period, then sample at $\frac{n}{T}$ to get the coefficients. We note, if $x(t)$ is real, then the fourier transform $X(u) = X^*(-u), c_n = c_n^*$.

Theorem 1 (Parseval's Theorem)**.**

$$\int_T |x(t)|^2 dt = T\sum_{-\infty}^{\infty} |c_n|^2$$

Gibbs said there exists convergence, but not in mean-suare error way:

$$\lim_{n\to\infty} X_N(t) \neq X(t)$$

but, $\int_T |x(t) - x_T(t)|^2 dt = 0$.

Example is two plots with same integral/ area, but one has discontinuities that prevent MSE convergence. We note that *Gibbs phenomenon* occurs at sharp discontinuities.

Theorem 2 (The Derivative Theorem)**.**

$$\frac{d}{dt}^n x(t) \leftrightarrow (j2\pi u)^n X(u)e^{j2\pi ut}$$

Proof.

$$x(t) = \int X(u)e^{j2\pi ut}du$$

let

$$y(t) = \frac{d}{dt} = \frac{d}{dt}\int X(u)e^{j2\pi ut}du$$

$$= \int X(u)(j2\pi u)e^{j2\pi ut}du = Y(u)$$

therefore, if

$$y(t) \leftrightarrow Y(u)$$

then

$$\frac{d}{dt}x(t) \leftrightarrow j2\pi uX(u)$$

$\square$

Theorem 3 (Scaling Theorem)**.** *assume* $a \neq 0$

$$x(at) \leftrightarrow \frac{1}{|a|}X(u/a)$$

Proof. let $\gamma = at$, then $d\gamma = adt$, $\frac{\gamma}{a} = t$, then

$$\frac{1}{a}\int x(\gamma)e^{-j2\pi\frac{u}{a}\gamma}d\gamma$$

$$= \frac{1}{|a|}X(\frac{u}{a})$$

□

we note that $|a|$ need for assumption in change of variables, accounts for $a > 0$ and $a < 0$.

1.3 Tuesday 9/3/2024

Definition 3 (Dirac Delta)**.**

$$\delta(t) = \lim_{A\to\infty} A * \frac{1}{A}$$

many ways to formulate: can also take the *sinc*. most important aspect is that area is 1.

Theorem 4 (Sifting Property)**.**

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$= \delta(t) * x(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau$$

Proof. $\delta(t)$ is even, therefore $\delta(t) = \delta(-t)$ and $\delta(t-\tau) = \delta(\tau - t)$. So integrating the product of $x(t)$ and $\delta(\tau - t)$, we get

$$\int x(t)\delta(t) = \int x(t)\delta(t-\tau)d\tau$$

$$= x(t)\int \delta(t-\tau)d\tau$$

$$= x(t)$$

□

Convolving the dirac delta with a signal gives the signal itself and allows extraction of a specific value. $\delta(t-\tau)$ when convolved with $f(t)$ gives $f(\tau)$. We tend to only have problems with $\delta(t)$ in integrations Some properties...

Definition 4 (Shift Property)**.** $x(t) * \delta(t-\tau) = x(t-\tau)$

1.3.1 Fourier Transform of dirac delta

$$\delta(t) \leftrightarrow 1 \tag{6}$$

This means that in convolution, the dirac delta is considered the identify function, so convolving, you're multiplying by 1 in the freqeuncy domain and the dirac delta is the identify in the spatial/time domain.

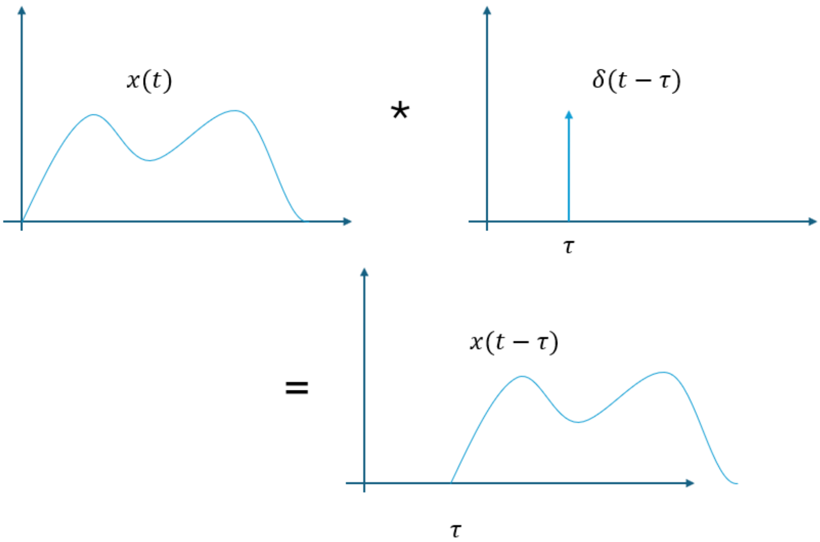

Figure 1: Shifting property of dirac delta.

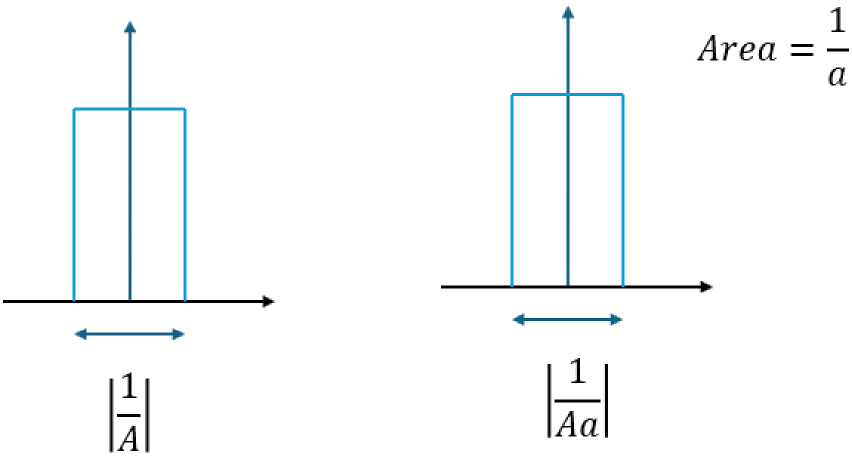

Figure 2: Scaling property of dirac delta.

1.3.2 Scaling

$$\delta(t) = |a|\delta(at) \tag{7}$$

2 for visualization

1.3.3 Duality

$$exp(j2\pi\xi t) \leftrightarrow \delta(u-\xi) \tag{8}$$

if we multiply a signal $x(t)$ by $exp(j2\pi fet)$, this is multiplying by a sinusoid, which gives a frequency shift, so the FT of this product is $X(u) * \delta(u-fe) = X(u-fe)$.

if $x(t)$ is real, then:

$$x(t)cos2\pi fet = x(t)\frac{1}{2}[exp(2\pi jfet) + exp(-2\pi jfet)] \leftrightarrow \frac{1}{2}X(u-fe) + frac12X(u+fe) \tag{9}$$

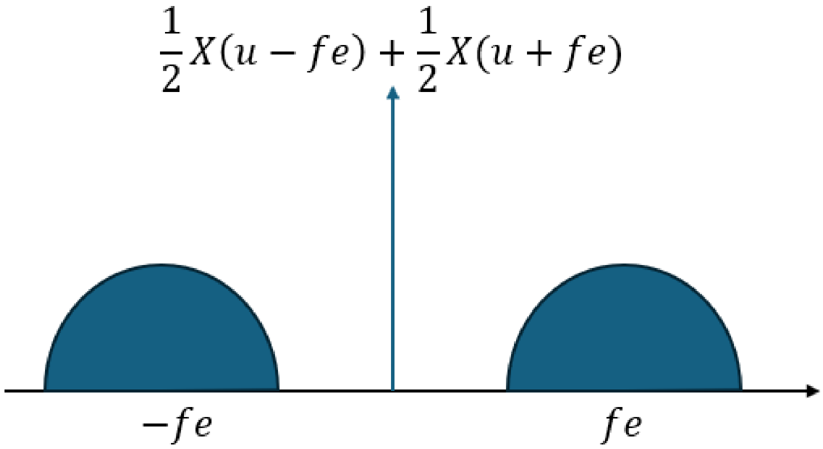

Figure 3: FT result if signal real.

3 for visualization.

If a signal is periodic, then it has fundamental frequency and harmonics, then the FT should get the same. We note that fourier transform is a linear operation. let's take the FT of a signal in it's fourier series representation:

$$Fx(t) = F\sum c_n exp(j2\pi u\frac{t}{T}) = \sum F[c_n exp(j2\pi u\frac{t}{T})]\text{by linearity of FT} = \sum c_n F[..e..]\text{by homogeneity} = \sum c_n \delta(u - \tag{10}$$

Another interpretation is **"centered under each of it's harmonics"**. Area under each δ is c_i see figure 4.

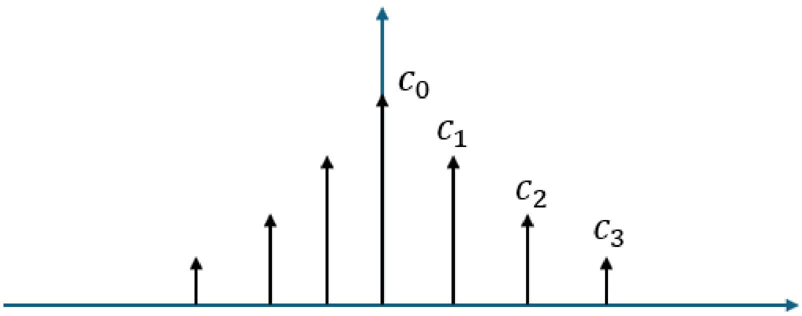

Figure 4: harmonics

Definition 5 (Sinc Function)**.**

$$sinc(t) = \frac{sin(\pi t)}{\pi t}$$

Definition 6 (kroenecker delta)**.**

$$\delta[n] = \begin{cases} 1, & if\, x = 0 \\ 0, & otherwise \end{cases}$$

the kroenecker delta is a sinc sampled at all $n \in \mathbb{Z}$.

Definition 7 (Rect Function $\Pi(t)$)**.**

$$\Pi(t) = \begin{cases} 1, & |t| < \frac{1}{2} \\ 0, & if|t| > \frac{1}{2} \\ \frac{1}{2}, & if|t| = \frac{1}{2} \end{cases}$$

if we move a single poitn on $x(t)$, we get the same fourier transform. Thus, we can reconstruct for many dislocated points. so how about at discontinuities? left continuous? right continuous? doing inverse FT on $X(u)$, you get disconinuity points in the middle. This is why for the rect function at $t = \frac{1}{2}$, we have $\Pi(t) = \frac{1}{2}$; isolated singular points.

$$\Pi(t) \leftrightarrow sinc(u) \tag{11}$$

1.4 Thursday 9/5/2024

Definition 8 (Sgn Function)**.**

$$sgn(t) = \begin{cases} -1; & t < 0 \\ 0; & t = 0 \\ 1; & t > 0 \end{cases}$$

$$sgn(t) \leftrightarrow \frac{1}{j\pi u} \tag{12}$$

Definition 9 (Unit Step Function, $\mu(t)$)**.**

$$\mu(t) = \frac{1}{2}(sgn(t) + 1)$$

Thus by linearity, we get:

$$\mu(t) \leftrightarrow \frac{1}{2}(\delta(t) - \frac{j}{\pi u}) \tag{13}$$

Definition 10 (Gaussian curve)**.**

$$f(t) = e^{-\pi t^2}$$

$$e^{-\pi t^2} \leftrightarrow e^{-\pi t^2} \tag{14}$$

Definition 11. *sequence of dirac*

$$\delta$$

s

$$comb(t) = \sum \delta(t - n)$$

convolve any single period and you get a periodic function

$$comb(t) \leftrightarrow comb(u) \tag{15}$$

note, there exist class of functions that FT to themselves.

Proof. comb is periodic, so can expand via fourier series

$$comb(t) = \sum_n c_n e^{j2\pi n \frac{t}{1}}$$

coefficient c_n is:

$$c_n = \frac{1}{1}\int_{-frac12}^{frac12} \delta(t)e^{-j2\pi nt}dt$$

now apply sifting property of δ

$$= exp(-2j\pi nt), t = 0 = 1$$

thus, $comb(t) = \sum_n c_n e$ as a result, $FT[comb(t)](u) = \sum_n F[e]$, and since $F[e] = \delta$,

$$F(comb) = \sum_n f(u-n) = comb(u)$$

□

We can use the comb to characterise periodic functions:

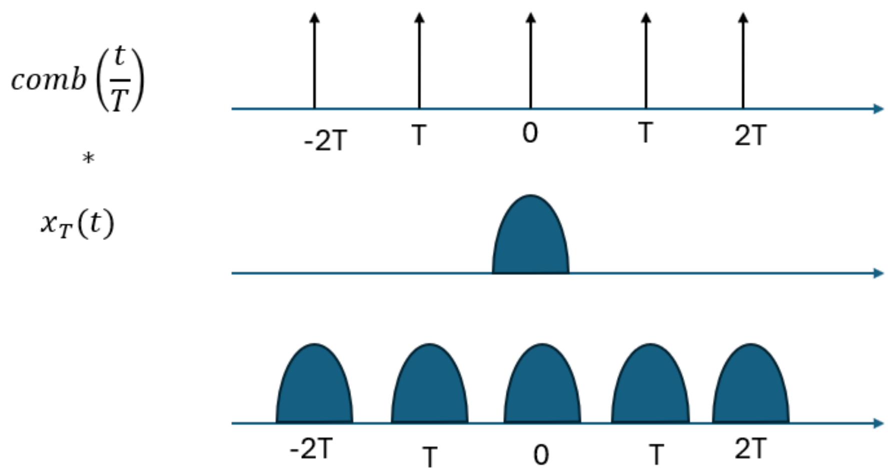

Figure 5: Using Comb to produce periodic function

$$x(t) = x_T(t) * \frac{1}{T}comb(fractT) \tag{16}$$

Definition 12 (Triangle Function Λ)**.**

$$\Lambda(t) = (1-|t|)\Pi(\frac{1}{2})$$

$$\Lambda(t) = \Pi(t) * \Pi(t) \leftrightarrow sinc^2(u) \tag{17}$$

Definition 13 (Array Function $array_m(t)$)**.**

$$array_m(t) = \frac{sin(\pi Mt)}{Msin(\pi t)}$$

Let's examine a finite comb:

$$\sum_{-M}^{M} \leftrightarrow \sum_{-M}^{M} e^{-j2\pi nu}$$

$$= 1 + 2\sum_{n=1}^{M} e^{-j2\pi nu} + e^{-j2\pi nu} \text{ by symmetry}$$

$$= 1 + s\sum_{1}^{M} cos(2\pi nu)$$

$$= 1 + 2Re\sum_{1}^{M} e^{2\pi nu} \quad (18)$$

in the last row, we have a geometric series, so we can rewrite in a closed form. combining iwth the definition of eulers, we get sin. thus,

$$\sum_{-M}^{M} \delta(t-n) \leftrightarrow (2M+1)array_{2M+1}(u) \quad (19)$$

we see that if we let $M \to \infty$, the left and right sides approach the comb function.

Definition 14 (Gamma function)**.**

$$\Gamma(\xi) = \int_{-\infty}^{\infty} \tau^{\xi-1} e^{-\tau} d\tau; Re(\xi) > 0$$

The gamma function belongs to group called "transcendental functions". For natural number $n \in \mathbb{N}$, this is factorial function: $\Gamma(n) = n!$. Can consider $\Gamma(\xi)$ to be partial factorial function.

$$\Gamma(\xi + 1) = \xi\Gamma(\xi) \quad (20)$$

Definition 15 (Bessel Functions $J_v(z)$)**.**

$$J_v(z) = \frac{2(\frac{z}{2})^v}{\sqrt{\pi}\Gamma(v+\frac{1}{2})} \int_0^1 (1-u^2)^{v-\frac{1}{2}} cos(zu)du, Re(v) > \frac{-1}{2}$$

a property is that if you take the fourier transform of a semi circle, you get something similar to a bessel function. A subset of bessel functions is the jinc function

Definition 16 (Jinc Function)**.**

$$jinc(t) = \frac{J_1(2\pi t)}{2t} = 2\int_0^1 \sqrt{1-u^2} cos(2\pi ut)du$$

we note the two parts of the integral: LHS is even, RHS is even

1.5 Tuesday 9/10/2024

When $v = 0$, we get:

$$J_0(2\pi t) = \frac{2}{\pi} \int_0^1 ... \quad (21)$$

for jinc, when $z = 2\pi t$, divide by $2t$.

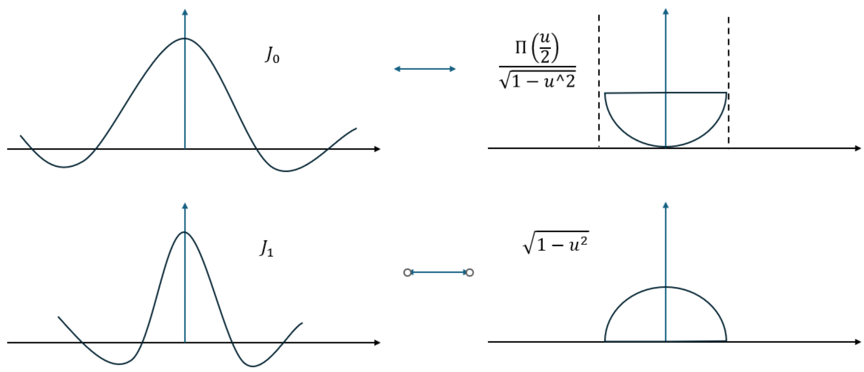

Figure 6: Jinc observations

Definition 17 (Sine Integral)**.**

$$Si(\Pi t) = \pi \int_0^t sinc(\tau) d\tau$$

we have to define this because there is no closed form solution note: $J_0(2\pi t) \leftrightarrow \frac{\Pi(\frac{u}{2})}{\sqrt{1-u^2}}$
weighted polynomials and hermite polynomials also functions that FT to themselves

Theorem 5 (Signal Integral Property)**.**

$$X(0) = \int_{t\in\mathbb{R}} x(t)dt,$$

Corrolary:

$$\int_{t\in\mathbb{R}} \frac{d}{dt} x(t)dt = 0 if X(0) < \infty$$

This follows directly from definition of FT:

$$\int_{-\infty}^{\infty} x(t)e^{-j2\pi ut}dt|_{u=0} = \int_{-\infty}^{\infty} x(t)dt \tag{22}$$

we note that if: $y(t) = \frac{d}{dt}x(t) \leftrightarrow Y(u) = j2\pi uX(u)$, then $\int_{-\infty}^{\infty} y(t)dt = Y(0) = 0 if X(y) < \infty$
Observe the conjugate symmetry of real signal spectrum

$$X(u) = X^*(-u) \tag{23}$$

we have a conjugate (hermitian) system if: $x(t) = x^*(-t)$. It is becaues of the conjugate symmetry of real signals that we don't have to worry about negative frequencies. If $x(t)$ is real, then $X(u)$ is conjugatively symmetric. In face, the real part is even, the imaginary part is odd.

$$X(u) = Re(u) + jIm(u) \tag{24}$$

$$X^*(-u) = Re(-u) + jIm(-u) \tag{25}$$

$$\rightarrow Re(u) = Re(-u), Im(u) = -Im(-u) \tag{26}$$

As polar coordinates, the magnitude function is even and phase components are odd.

Theorem 6 (Power Theorem)**.**

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(u)Y^*(u)du$$

a special case is if $x(t) = y(t)$. we note that $|z|^2 = zz^*$. therefore:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(u)|^2 du \tag{27}$$

the above is also parseval's theorem

Proof.

$$\begin{aligned}\int_{-\infty}^{\infty} x(t)y(t)dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)e^{2j\pi ut} du y(t)dt \\ &= \int_{-\infty}^{\infty} X(u)\int_{-\infty}^{\infty} y^*(t)e^{2j\pi ut} dtdt \\ &= \int_{-\infty}^{\infty} X(u)(\int_{-\infty}^{\infty} y^*(t)e^{\cdots} dt)^* du \\ &= \int_{-\infty}^{\infty} X(u)Y^*(u)du\end{aligned}$$

□

Let's examine $J_0(2\pi t) \leftrightarrow \frac{\Pi(\frac{u}{2})}{\sqrt{1-u^2}}$ What's the energy of $J_0(2\pi t)$? By power theorem,

$$E = \int_{-1}^{1} \frac{du}{1-u^2} du = \infty \text{ by parseval's theorem} \tag{28}$$

Another example:

$$\int sinc^2(t)dt = \int \Pi^2(u)du = \int \Pi(u)du = 1 \text{ by parseval's} \tag{29}$$

Definition 18 (Poisson sum formula)**.** *Assuming a period T,*

$$T\sum_{n=-\infty}^{\infty} x(t-nT) = \sum_{n=-\infty}^{\infty} X(\frac{n}{T})e^{j\frac{2\pi nt}{T}}$$

The idea for the proof is to take $x(t)$ and repeat it over intervals of length T. Because this is periodic, thus have a fourier series representation.

Proof. ... □

Theorem 7 (LTI Frequency Response theorem)**.**

$$x(t) * e^{j2\pi vt} = X(v)e^{j2\pi vt}$$

Example 1. *Let $x(t) = e^{j2\pi vt}$ for a fixed v fixed. we send $x(t)$ through an LTI system with an impulse response of $h(t)$ to get $y(t) = x(t) * h(t) = e^{j2\pi vt} * h(t)$.*

$$\begin{aligned}y(t) &= \int e^{j2\pi v(t-\tau)}h(\tau)d\tau \\ &= e^{2\pi vt}\int e^{-j2\pi v\tau}h(\tau)d\tau \\ &= e^{2\pi vt}H(v) \text{ } H \textit{ being the freq response of the LTI filter}\end{aligned}$$

We've thus observed that the input freq of v, output freq of v, weighted by H(v), which is complex. due to an LTI system, which is noncreative.

so the problem with an impulse function $\delta(t)$ is infinite energy. So what do you do?

1.6 Tuesday 9/17/2024

Other types of FT

Definition 19 (Discrete Time Fourier Transform)**.**

$$X_D(f) = \sum x[n]e^{-j2\pi nf}$$

$$x[n] \int X_D(f)e^{j2\pi nf}df$$

Fourier transform goes from continuous to continuous. DTFT is discrete to continuous. Fourier series goes from continuous to discrete. DFT (or FFT) goes from discrete to discrete.

Example 2. *Let's example* $X_D(f) = \sum x[n]e^{-j2\pi nf}$
claim: periodic with a period of 1.

$$X_D(f-1) = \sum_n x[n]e^{-j2\pi n(f-1)}$$

$$= \sum_n x[n]e^{-j2\pi nf}$$

therefore, $X_D(f) = X_D(f-1)$ *note, similar to fourier series but in freq domain.*

similar to how the dirac delta is a unit operation in convolution, the kroenecker delta is unit operation in discrete. this exists and can be implemented. we reiterate that the dirac delta does not due to infinite energy.

2 Ch5: Sampling theorem

2.1 Thursday 9/19/2024

Sampling theorem. developed to show we could use discrete. **Sampling theorem is dual of fourier series!**

theorem says if bandlimited, we can get back original signal. Need T to be smal lenough so $\frac{1}{T}$ far enough; hence why we have nyquist rate $\frac{1}{T} > 2B$ required.

Theorem 8 (Cardinal series)**.**

$$x(t) = \frac{1}{\pi}\sum_n x(\frac{n}{2B})\frac{sin(\pi(2Bt-n))}{2Bt-n}$$

Because a signal is bandlimited in low-pass sense, we can say

$$X(u) = X(u)\Pi(\frac{u}{2B}) \tag{30}$$

ie identically zeros for $|u| > B$.

Theorem 9 (Sampling theorem)**.**

$$x(t) = \sum_{-\infty}^{\infty} x(\frac{n}{2B})sinc(2Bt-n)$$

Note that for $t = 0$, only one sinc contributes to the overall signal, therefore it does not contribute to the other sampling points.

The problem with sampling theorem is the need for infinite samples, although we technically don't need that many.

bandlimited signals are not causal. causal is not bandlimitedbut conundrums exist.

$$E = \int |x(t)|^2 dt = T \sum_n |c_n|^2 = T \sum_n |x(nT)|^2 \tag{31}$$

if you have a finite energy signal, this is time bandwidth product. this determines complexity of a signal.: 2BT over time interval T.

If we lose a sample, how can we recover it?

Example 3.

$$\begin{aligned} y(t) &= r \sum_{n \neq 0} x(\frac{n}{2W}) sinc(2Bt - rn) \\ &= r \sum_n x(\frac{n}{2W}) sinc(tB2 - rn) \\ &= x(t) - rx(0) sinc(2Bt) \end{aligned}$$

This gives frequency lowered. We can only do this if oversampled.

Works in practice, but if you lose too much, noise in real life will make it unrecoverable. This is an ill-conditioned problem. Increasing lost samples leads to more instability.

2.2 Thursday 9/26/2024, Tuesday 10/1/2024

Extension to multidimensional version. Have the 1-D properties with some extra ones:

- separability
- rotation, scale, transposition

Definition 20 (scale)**.**

$$x(A\vec{t}) = \frac{A^{-T}\vec{u}}{|detA|}$$

When A has specific properties, we get rotation, scale, and transposition. In general, when performing higher dimension fourier transforms, we do 1D FT one direction at a time; this is due to separability of $e^{\cdots}$.

A comment the FFT is limited. let's look at continuous time. requires even sampling.

Other concepts: shift, mechanics of convolution, the separability;

$x(\vec{t}) = \Pi x_i(t_i) \leftrightarrow \Pi X_i(u_i) = X(\vec{u})$

2.3 Thursday 10/3/2024

last time, separability:

$$x(\vec{t}) = \Pi_n x_n(t_n) \leftrightarrow X(\vec{u}) = \Pi_n X_n(u_n) \tag{32}$$

although functions may not be separable, if we can show a function is sum of separable functions, we can still do the same.

$$x(t_1, t_2) = \Pi(\frac{t_1}{c} - \frac{1}{2})\Pi(\frac{t_2}{c} - \frac{1}{2}) + \Pi(\frac{t_1}{c} + \frac{1}{2})\Pi(\frac{t_2}{c} + \frac{1}{2}) = \Pi(\frac{t_1}{c} \pm \frac{1}{2})\Pi(\frac{t_2}{c} \pm \frac{1}{2}) \tag{33}$$

Not all functions are separable, but we can write them as sums of separable functions.

$$x(\vec{t}) = \sum_m \prod_n^N x_m^n(t_n), \text{N dimensions} \tag{34}$$

$$x(\vec{t}) \leftrightarrow \sum_m \prod_n^N X_m^n(u_n) \tag{35}$$

We ask, $Ax(\vec{t}) = \vec{\tau}$, what is $x(A\vec{t})$ with matrix A? we get a generalization of scaling theorem: $x(at) = \frac{1}{|a|}X(\frac{u}{a})$. We have three types of matrices A:

- **transposition**, flipping over an axis:A= $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$
- **scale**, stretch along an axis: A= $\begin{pmatrix} \frac{1}{M_1} & 0 \\ 0 & \frac{1}{M_2} \end{pmatrix}$
- **rotation**, maintaining size and shape: A= $\begin{pmatrix} cos(\theta) & sin(\theta) \\ \sin(\theta) & cos(\theta) \end{pmatrix}$

couple comments. for transposition and rotation, $det(A) = 1$. when this holds, $A^{-T} = A$. so when are these operations commutative?

2.4 Tuesday 10/18/2024

we know that for scaling: $x(A\vec{t}) \leftrightarrow \frac{A^{-T}\vec{u}}{|detA|}$ due to rotation having det of 1, we can say: $x(R\vec{t}) \leftrightarrow X(R\vec{u})$.

2.4.1 circular symmetry

when we have: $x(\vec{t}) = f(r)$. this is a reduction to 1 dimension. We would think it means FT of 1D is also 1D. however, we need to account for change of variables.

$$x(\vec{t}) = f(r) \leftrightarrow \int_0^{2\pi} \int_0^{\infty} f(r)exp(j2\pi r\rho cos(\theta - \phi))rdrd\theta \tag{36}$$

We get a new fourier kernel here:

$$exp(-j2\pi(u_1t_1 + u_2t_2)) = exp(-j2\pi(r\rho cos(\theta)cos(\phi) + r\rho sin(\theta)sin(\phi)))$$

$$= exp(-j2\pi r\rho(cos()cos() + sin()sin()))$$

So we have: $f(r) \leftrightarrow F(\phi)$.

look at cosine. cosine periodic, so $f(cos(\theta))$ also periodic. Thus,

$$\int_0^{2\pi} = \int_{0+\phi}^{2\pi+\phi}$$

and therefore

$$\int_{r=0}^{\infty} f(r)[\int_{\theta=0}^{2\pi} e^{\dots\theta-\phi\dots}d\theta]rdr$$

$$\int_{r=0}^{\infty} f(r)[\int_{\theta=0}^{2\pi} e^{-j2\pi r\rho cos\theta}d\theta]rdr$$

Inner integral is a constant. we end up with bssel function of 0th order. Therefore, due to circular symmetric functions we have a special fourier transform

Definition 21 (Fourier-Bessel/Hankel Transform)**.** *For a function $f(\vec{t})$ that can be written as a circular function $f(r)$,*

$$f(r) \leftrightarrow F(\rho) = 2\pi \int_0^\infty r f(r) J_0(2\pi r\rho) dr$$

Example 4. *let's look at a trussel (circle with circle inside removed). we have two ways to evaluate the FT:*

$$X(u_1, u_2) = \int\int (...)$$

or

$$F(r) = 2\pi \int_0^\infty \delta r J_0(2\pi r\rho) dr$$

Both are identical. in most cases, Hankel transform is easier. However, the Gaussian function is an exception due to its separability.

Example 5. *FT of a circle radius 1: $f(r) = \Pi(\frac{r}{2})$*

$$f(r) \leftrightarrow F(\rho) = 2\pi \int_0^\infty r f(r) J_0(2\pi r\rho) dr$$

$$= 2\pi \int_0^1 r J_0(2\pi r\rho) dr$$

because we know $\frac{d}{da} a J_1(a) = a J_0(a)$, therefore

$$F(\rho) = 2jinc(\rho)$$

In general, if we have $f(\frac{r}{m}) \leftrightarrow m^2 F(m\rho)$. Note the m^2; this maintains dimensionality since $x(\frac{t_1}{m}, \frac{t_2}{m}) \leftrightarrow m^2 X(mu_1, mu_2)$.

2.5 Thursday 10/10/2024

tomography projection: fourier slice theorem. we define projection as:

Definition 22 (projection)**.**

$$p(t_1) = \int_{-\infty}^\infty x(t_1, t_2) dt_2$$

Let's take the 1D Ft of the projection:

$$P(u_1) = \int_{-\infty}^\infty p(t_1) e^{-j2\pi u_1 t_1} dt_1$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty x(t_1, t_2) e^{-j2\pi u_1 t_1} dt_1 dt_2$$

$$= \int_{-\infty}^\infty \int_{-\infty}^\infty x(t_1, t_2) e^{-j2\pi(u_1 t_1 + u_2 t_2)} dt_1 dt_2 \text{ if } u_2 = 0$$

$$= X(u_1, u_2 = 0)$$

Therefore, we see the FT of projection vertically is the central slice of a 2D FT (horizontally). We have thus shown it is sufficient to take projections to reconstruct $X(u_1, u_2)$.

Let's discuss back projections. There exists emphasis on low frequency the closer you are to the center. This also means we miss higher frequencies.

We examine the radon transform and its inverse.

Definition 23 (radon transform)**.**

$$p_\theta(t_1) = \int x(R_\theta \vec{t})dt_2$$

$$P_\theta(u_1) = X(u_1 cos\theta, u_1 sin\theta)$$

therefore,

$$x(t_1, t_2) = \int_0^{2\pi} \int_0^{\infty} X(\rho cos\phi, \rho sin\phi)e^{-j2\pi\rho(t_1 cos\phi + t_2 sin\phi)} d\rho d\phi$$

The derivative is similar to taking high-pass filter (ie ramp filter), so we have to be careful. filtered backprojection is the hilbert transform of the derivative.

$$\hat{P}_\phi(t) = \frac{1}{2\pi^2 t} * \frac{dp_\phi(t)}{dt} \tag{37}$$

Note, we dont have ot worry about smapling rate for images, appleid to number of projections.

2.6 Thursday 10/24/2024

2.6.1 Periodicity in N-Dimensions

let's take idea of a period and extend it to M-D. we can talk about the "center" of a period. recognize through periodicity vector:

Definition 24 (Periodicity Vector)**.**

$$P = [\vec{P}_1 | \vec{P}_2 ... | \vec{P}_N]$$

Periodicity vectors are not unique.

Definition 25 (periodic function in N-D)**.**

$$S(\vec{u}) = s(\vec{u} - P\vec{m})$$

with the smallest possible $|detP|$.

With any periodic function, there exists a tile. one example is a rectagular tile with shapes inside. The tile can always be the parallelogram associated for each vector. All possible tiles have the same area. Need not be completely jo...

Definition 26 (Multidimensional Fourier Series)**.** *let $Q^T = P^{-1}$.*

$$S(\vec{u}) = \sum_{\vec{m}} c[\vec{m}]exp(-j2\pi\vec{u}^T Q\vec{m})$$

$$st\ c[\vec{m}] = |detQ| \int_{\vec{u} \in C} S(\vec{u})exp(j2\pi\vec{u}^T Q\vec{m})d\vec{u}$$

for a tile C. this integral is equivalent to integrating over a single period. denote "cell" to generalize shape.

Example 6.

$$P = \begin{pmatrix} 6 & 3 \\ 0 & 5 \end{pmatrix}$$

$$Q = P^{-T} = \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{-1}{10} & \frac{1}{5} \end{pmatrix}$$

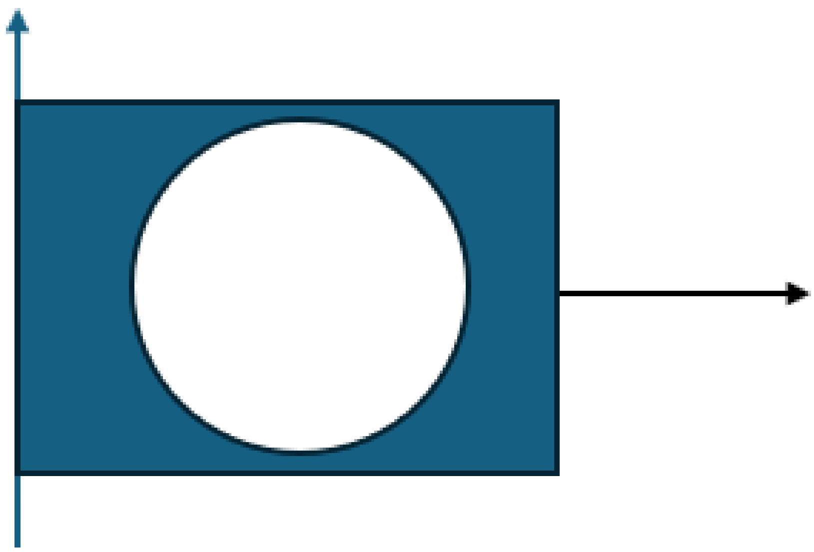

Figure 7: One cell/period

,

$$Q\vec{m} = \begin{pmatrix} \frac{m_1}{6} \\ \frac{-m_1}{10} + \frac{m_2}{5} \end{pmatrix}$$

Thus the fourier coefficient is:

$$|detP|X(\vec{u}) = \Pi(\frac{u_1 - 3}{6})\Pi(\frac{u_2}{5}) - \Pi(\frac{\sqrt{(u_1-3)^2 + u_2^2}}{4})$$

Left part is a shifted rect, right part is a shited and scaled circle. FT transform give:

$$|detP|X(\vec{u}) \leftrightarrow |detP|x(\vec{t})$$

$$= [30sinc(6t_1)sinc(5t_2) - 8jinc(2\sqrt{t_1^2 + t_2^2})]exp(-j6\pi t_1)$$

$$\therefore c[\vec{m}] = |detQ| \int_{u \in C} S(\vec{u})exp(j2\pi u^T Q\vec{m})d\vec{u}$$

Under the integral, it is the inverse FT, sampled at a $Q\vec{m}$ rate.

$$\therefore c[m_1, m_2] = (-1)_1^m(\delta[m_1]sinc(\frac{m_1}{2} - m_2) - \frac{4}{15}jinc(2\sqrt{(\frac{m_1}{6}^2 + (\frac{m_1}{10} + \frac{m_2}{5})^2}))$$

note in 1D, if x(t) is real, then $X(u) = X^*(-u)$. so why not half the sample? this doesn work because if we half it, then it is no longer a real signal.

If you take the phase and magnitude of FT, what's more important? setting magnitude to 1 (similar to performing a highpass filter), we get an edge depiction of the image. if we set phase to 1, we get a bad image. in terms of preserving image, phase allows outline of images. periodicity in the freq corresponds to sampling in time domain.

Definition 27 (sampling, replication matrix)**.** *If Q is a sampling matrix, then P is a replication matrix.*

$$Q = P^{-T}$$

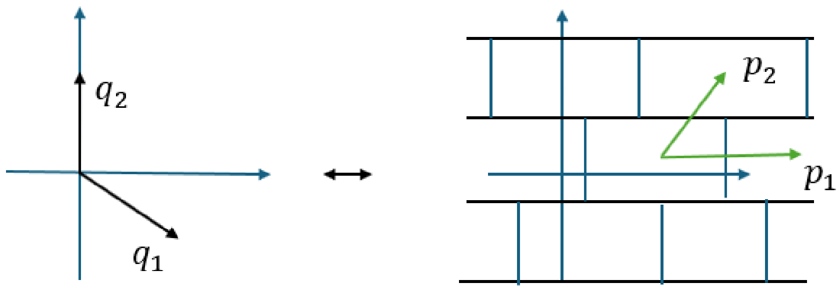

Figure 8: Visuzalization of Sampling and Replication Matrices

Theorem 10 (Multidimensional Comb Lemma)**.** *for a sampling matrix Q and its corresponding periodicity matrix $P = Q^{-T}$:*

$$\sum_m \delta(\vec{t} - Q\vec{m}) \leftrightarrow |detP| \sum_m \delta(\vec{u} - P\vec{m})$$

As P spreads out, Q shrinks, leading to more samples per unit area. We want to pick P so Q is as far apart as possible. So what is the sampling density?

If we assign one sample per tile, then

$$\frac{1}{|detA|}\text{samples per unit area}$$

$$= |detP| = sampledensity(SD)$$

We want SD as large as possible with aliasing; equivalent to the nyquist theorem.

There is a difference between the nyquist rate and the minimal sampling rate. In Nyquist, we can recover lost signals. Nyquist is the minmal packing. The minimum sampling rate is the minimum are a of the sample (ie. the ellipse in a square). The zeros in between the nyquist rate can be used to recover missing samples. The minimum sampling is much more complex to use. alos, we can't have infinite signal; instead, we look at energy

$$E = \int |x(t)|^2 dt = \frac{1}{2B} \sum |x(\frac{n}{2B})|^2$$

The above is simply another version of parseval's theorem.

This is approximately $\frac{1}{2B} \sum_{-N}^{N} |x(\frac{n}{2B})|^2$; certain time intervals T gives 2BT samples needed.

Theorem 11 (MD Sampling theorem)**.**

$$x(t) = \sum_m x(Q\vec{m}) f_c(\vec{t} - Q\vec{m})$$

f_c is the interpolation function that has a FT $f_c(\vec{t}) \leftrightarrow \frac{\Pi_c(\vec{u})}{|detP|} = |detQ| \int_{u \in C} e^{-j2\pi ut} dt$

Some comments. support of maximally packed circles is a heaxagon. If samples are with a quare: $SD_{rect} = 4W^2$ area of tile. using hexagon, we get a smaller area $SD_{nyq} = 2\sqrt{3}W^2$.

$$r_2 := \frac{SD_{nyq}}{SD_{rec}} \approx .866(\text{fewer samples per area})$$

For higher dimensions N, R_N decreases. The FT in 1 direction of 2D freq will get 1D signals that are bandlimited.

3 Ch11: POCS

Definition 28 (convex set)**.** *S convex if* $\forall x, y \in S, \forall \lambda \in [0, 1]$

$$\lambda x + (1 - \lambda) y \in S$$

Examples of convex sets:

- **bounded**: $C : x[n] | 0 \leq x[n] \leq u$
- **identical middles**: $C = x(t) | x(t) = c(t); t \in T$
- **fixed area signals**: $C = x(t) | \int_{-T}^{T} = A$
- **bounded signals**: $C = x[n] | 0 \leq x[n] \leq u$
- **identical middles**: $x[n] | x[3] = 1$. When $n = 3$, this becomes a plane.
- **bounded energy signals**: $x(t) | \int |x(t)|^2 dt \leq E$, reminder that $\int |x(t)|^2 dt = |x(t)|^2$
- **band limited signals** $x(t) | X(u) = 0; u > B$
- **tomographic projections** $C_p = f(x, y) | \int f(xmy) dx dy = p$

Comment: connection to hilbert space, in terms of FT: the FS is a point in the space.
Now discussion of projecting onto these sets.

Definition 29 (projecting onto a set)**.** *Assume we have a point p and a convex set C. The* ***projection*** *of p onto a set C is: p if* $p \in C$*, or* $p' \in C$ *such that* $argmin_{p'} |p - p'|_2$

due to convexity, this point is unique. because the projection of a point on the set is the point itself, the projection function is idempotent.

Definition 30 (idempotent)**.** P_C *projection is* ***idempotent*** *if* $P_C(P_C(x)) = y$ *if* $P_C(x) = y$

POCS is projecting between two sets until you find a fixed point that belongs to all the sets.

3.1 Tuesday 11/12/2024

projecting an arbitrary signal onto set of bandlimited signals: simply apply lowpass filter.
projecting onto set of fixed energy is similar to projecting onto set where phases is the same!

Lemma 1 (pocs all intersecting)**.** *alternating between N convex sets with intersections will converge to common point.*

Definition 31 (Von Newmann's alternating Projection Algorithm)**.** *consider 2+ intersecting planes. always projects to poitn nearest to the intersection. each projection in each iteration forms a plane.*

Lemma 2 (pocs nonoverlapping)**.** *POCS converges to point closest to each of the sets. forms a* ***limit cycle****.*

for multiple sets, the limit cycle is not unique. there are workarounds for this. one is weighting each projection.

Definition 32 (Papoulis Gerchberg Algorithm)**.** *restoration of lost data.*

Set 1:identical tails

Set 2: bandlimited

LP filter to project onto set 2, then replace tails to project onto set 1, and repeat.

if the tails are analytic, then there is only 1 solution for convergence. in general, we look at eigenvalues to determine sensitivity. the P-G algorithm is ill-posed and ill-conditioned.

Some other examples: **Neural Network Associative Memory**. Filling an image or completing an image. The problem with associative memory is that it's not very good. Can result in a limit cycle of image searched for if not in original.

Combining low-res sub-pixel shifted images. Overcoming problem of low-res, combining multiple low res parts. Note, it forms a convex set.

The problem with POCS:

- sets of high resolution
- positivity
- resolution limit

Bertrand's paradox: taking random chord in circle, probability of the length being a certain length. There are different solutions depending on how the chord is randomly selected. This leads us to the realization that there is no paradox but that we need to more clearly define what is random.